Geometric interpretation of Tensor-Vector-Scalar theory in a Kaluza-Klein reference fluid

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Abstract. Gravitational alternatives to dark matter require additional fields or assumptions beyond general relativity while continuing to agree with tight solar system constraints. Modified Newtonian Dynamics (MOND), for example, predicts the Tully-Fisher relation for galaxies more accurately than dark matter models while limiting to Newtonian gravity in the solar system. On the other hand, MOND does a poor job predicting larger scale observations such as the Cosmic Microwave Background and Matter Power Spectra. Tensor-Vector-Scalar (TeVeS) theory is a relativistic generalization of MOND that accounts for these observations without dark matter. In this paper, a generalized TeVeS from Kaluza-Klein theory in one extra dimension is derived as a consequence of $n = 0$ Kaluza-Klein modes. In the KK theory, MOND is a special case of a slicing condition in the 5D ADM formalism enforced by a reference fluid as in the Isham-Kuchař method which may arise from a broken displacement symmetry. This has two benefits: first is means that TeVeS is compatible with Kaluza-Klein dark matter theory, which is a strong candidate for Weakly Interacting Massive Particles (WIMPs), the other is that it provides an elegant mechanism for the scalar and vector fields. It constrains most of the freedom in the definition of TeVeS which does not have a field theoretic motivation. This is important because the Kaluza-Klein theory predicts that spin-2 tensor modes must propagate at the speed of light, in agreement with observation, from theoretical constraints while TeVeS has to match this observation empirically. Furthermore, it removes need for the interpolating function in MOND and the Lorentz-violating condition on the vector field to be physical since they are analogous to a gauge condition and depend on state of motion.

1. Introduction

Fritz Zwicky first proposed dark matter to explain the rotational curves of stars in the outer reaches of galaxies which appear to rotate far faster than Newtonian physics predicts [1]. The best evidence for Dark Matter, however, is at the largest scales. The ΛCDM model does an excellent job fitting observations at the cosmological scale as well as weak and strong lensing of galaxy clusters. At smaller scales, however, it runs into difficulty. Simulations and observations of individual galaxies (those with high mass and dwarf satellites in the Local Group) have led to the search for ways to either supplement or replace CDM at those scales [2].

In 1983, Milgrom proposed an alternative explanation for galactic rotation curves, modifying the Newtonian force at very low accelerations so that rotational curves match the Tully-Fisher relation [3] which is an empirical fitting to these curves [4][5]. The following year, Milgrom and Bekenstein published a modified Newtonian theory that became known as Modified Newtonian Dynamics or MOND [6]. Forty years later, MOND continues to be a serious contender to explain at least some dark matter attributed observations. A recent review of dark matter theories including MOND can be found in [7]. MOND has been shown to fit not only galactic rotation curves [8], better than halo models but also dwarf galaxies such as a recent study in the Fornax cluster [9].

MOND is a modification of the force a body experiences under gravity from the standard $F = ma$ to $F = m\mu(a/a_0)a$ such that $\mu(x) \to 1$ for $x \gg 1$, leading to Newtonian dynamics for the solar system as well as clusters of stars that are small enough where gravitational acceleration is significant, and $\mu(x) \to x$ for $x \ll 1$ which applies to galaxies. The parameter a_0 determines the acceleration at which MOND takes over from Newton.

Both dark matter and MOND fit the available data poorly in some arenas, which is why a hybrid approach may be best. Superfluid dark matter is another example of a hybrid theory that seeks to explain areas where dark matter fits the data well such as in weak lensing, galaxy clusters, and at the cosmological scale and at the galactic rotational curve level where MOND is a better fit [2].

Examples of areas where MOND is a poor predictor versus dark matter include colliding galaxy clusters such as the Bullet Cluster (1E 0657-558) and galaxy formation in the early universe. Early criticism of MOND that it was unable to reproduce asymmetric weak lensing and that, therefore, such lensing was direct evidence of dark matter were based on the spherically symmetric version [10]. Non-spherically symmetric MOND which do not neglect the additive curl field can reproduce nonlocal weak lensing of galaxy clusters [11][12]. When accelerations are close to or higher than a_0, for example Abell 1689 [13], behavior is highly sensitive to the interpolating function μ or MOND predicts behavior should be Newtonian while lensing shows it is not.

At the galactic scale, MOND does a much better job fitting data than non-interacting dark matter theories. MONDian theories can account for a great deal of the phenomena attributed to such particles at the galactic scale without the need for dark matter halo formation [14]. Halo density profiles are fitted to rotation curves for each galaxy but struggle to explain the lack of variation in rotational curves, especially at high mass scales. Simulations show a much larger scatter around the Tully-Fisher relation than observed [15]. For this reason, one would prefer not to give up on MOND completely at the galactic scale but clearly additional features are needed to account for large scale phenomena.

While superfluid dark matter theory achieves this hybrid theory via a state transition in dark matter, this paper shows how MOND arises from $n = 0$ modes of a Kaluza-Klein theory in one extra dimension. Dark matter, meanwhile, arises from $n = 1$ modes, which are the lightest Kaluza-Klein particles.

I achieve this by connecting the Kaluza-Klein theory to a relativistic generalization that is compatible with tight constraints on GR in the Solar System: generalized Tensor-Vector-Scalar (TeVeS) theory which is a generalization of a relativistic realization of MOND, Tensor-Vector-Scalar (TeVeS), [16] [17].

TeVeS succeeds in two important observations that Bekenstein's original TeVeS theory did not agree with: the propagation speeds of spin-2 tensor modes [18][19] and Baryonic Acoustic Oscillation peaks in the early universe [20] [17]. Indeed, Skordis et al. made a further generalization of TeVeS, coining it Aether-Scalar-Tensor theory or AeST, showing good agreement with all Cosmic Microwave Background (CMB) power spectra peaks [21]. Dustlike evolution of the scalar field in the early universe, which decouples from the vector field, has a similar effect on BAO as dark matter.

Generalized TeVeS remains a strong contender for an alternative to dark matter, partly because it introduces sufficient degrees of freedom to mitigate the need for it in some cases. It also shares features with Einstein-Aether theory that may address galaxies clusters including the bullet cluster [22].

In generalized TeVeS, as in the original TeVeS, the Bekenstein-Sanders metric [16] is disformal and universally couples to matter,

$$g^{ab} = e^{2\phi}\tilde{g}^{ab} + 2\sinh(2\phi)\beta^a\beta^b, \tag{1}$$

where $\tilde{g}^{ab}$ is the Einstein metric, ϕ is the scalar field, and β^a is the vector field (sometimes also given the symbol A^a or $\mathfrak{U}^a$). Bekenstein, building on Sanders [23], demonstrated that this is an invertible metric [16]. This one of the simplest disformal metrics and is invertible. Each of these has its own action as described in section 4 as well as constraints on ϕ given by μ and a constraint on β^a such that $\beta^a\beta_a = -1$. Only certain constraints, in a quasi-static approximation, give the correct MOND and Newtonian limits.

An open problem for TeVeS is that it is an *ad hoc* empirical theory, constructed to match observations. Its scalar field is constrained to obey MONDian physics. Likewise, its vector field is constrained in such a way as to violate Lorentz covariance. This constraint is relatively mild and appears to be related to the arrow of time, but it is unclear why this particular spin-1 field would obey such a constraint while others do not.

Since the actual form of the theory is not known from first principles or other observations but rather fit to the data, a field theoretic foundation is desired. For example, the form of μ is undetermined and different interpolating functions between the Newtonian and MOND regimes generate different results, particularly when a large percentage of matter experiences accelerations on the order of a_0.

In this paper, a mechanism by which TeVeS arises from Kaluza-Klein theory is proposed. In one extra dimension, additional fields take the form of a scalar lapse function and a vector shift function. Coordinate constraints are enforced via the reference fluid method of Kuchař and Torre [24]. This method proposes that spacetime events are marked by a physical reference fluid representing spacetime in the fifth dimension. This fluid has a stress-energy tensor in the Einstein-Field equations and thus, interacts with the gravitational field rather than being a passive marker of it.

The reference fluid theory is known to be dual to broken displacement symmetry at the quantum level, meaning that a theory with a reference fluid is the classical limit of a quantum gravity theory with a broken symmetry [25].

While the representation of time as a physical reference field is not new (see [26]), the application to the fifth dimension in a Kaluza-Klein theory *instead of time* is novel. Invoking the Arnowitt-Deser-Misner (ADM) [27] formalism, it is shown that, not only does Kaluza-Klein theory agree with TeVeS, it predicts that spin-2 tensor modes propagate at the speed of light as demonstrated in observations of GW170817 [28], a necessary but otherwise *ad hoc* modification that had to be made to the original TeVeS in order to agree with observation [17].

Define a $4+d$ dimensional metric, γ_{AB} where capital Latin letters refer to indexes $A, B = 0, 1, 2, 3, \ldots, 4+d-1$, lower case Latin letters refer to $4+d$ dimensional indexes $a, b, c = 0, 1, 2, 3, \ldots, 4+d-2$ where x_{4+d-1} is the ADM flow dimension rather than x_0 as in the standard formalism. From this theory arise a vector and scalar field which under certain conformal transformations, couples to matter in nearly the same way as TeVeS which becomes identical in both the quasi-static limit which leads to MOND as well as perturbations against an FLRW background for spin-2 tensor modes.

2. The ADM Formalism and Kaluza-Klein Slicing

In this section, some background on the ADM formalism is provided and it is shown how it can be applied to Kaluza-Klein theory.

2.1. ADM Formalism

The following is a short background on the ADM formalism largely derived from the lecture notes of Edward Bertschinger [29]. The ADM formalism has historically been designed to model the evolution of space over time in a Hamiltonian formulation. In cases of numerical relativity and canonical quantization the Hamiltonian is preferred. Hamiltonian field theory, as in mechanics, requires a preferred dimension of time.

Typically, in the case of, say, a relativistic particle, proper time is the clear choice. In field theory, however, time must be defined everywhere, requiring a separation of space and time coordinates. The resulting equations are valid for any choice of time coordinate but are not covariant .

The ADM formalism breaks up a metric g_{ab} on a $d+1$ dimensional manifold into three fields: the "spatial" metric γ_{ij}, the lapse function ξ, and the shift function β_i. This is valid for any metric. The Einstein-Hilbert action $\int d^{d+1}x\,\sqrt{-g}R$ is rewritten as the ADM action

$$\int d^{d+1}x\,\mathcal{L}_{ADM} \tag{2}$$

where $\mathcal{L}_{ADM} = \xi\sqrt{\gamma}[K_{ij}K^{ij} - K^2 + {}^{(d)}R]$ and neglecting boundary terms from total derivatives. In this case the extrinsic curvature satisfies

$$K_{ij} = \frac{1}{2\xi}(\nabla_i\beta_j + \nabla_j\beta_i - \partial_t\gamma_{ij}), \tag{3}$$

and $K \equiv \gamma_{ij}K^{ij}$. It then derives from the Einstein-Hilbert action a set of evolution equations and Hamiltonian constraints in the chosen time coordinate. The Hamiltonian has conjugate variables π^{ij} and γ_{ij} where the former is defined as $\pi^{ij} \equiv \frac{\partial\mathcal{L}_{ADM}}{\partial\dot{\gamma}_{ij}} = \sqrt{\gamma}(K\gamma^{ij} - K^{ij})$.

The evolution equations are then derives from the Hamiltonian which is given by,

$$H(\xi,\beta_i,\gamma_{ij},\pi_\xi,\pi^i,\pi^{ij}) = \int d^dx\,\left(\dot{\xi}\pi_\xi + \dot{\beta}\pi^i + \xi\mathcal{H} + \beta_i\mathcal{H}^i - \mathcal{L}_M\right),$$

$$\mathcal{H} = \frac{1}{\sqrt{\gamma}}\left(\gamma_{ik}\gamma_{jl} - \frac{1}{2}\gamma_{ij}\gamma_{kl}\right)\pi^{ij}\pi^{kl} - \sqrt{\gamma}^{(d)}R,$$

and

$$\mathcal{H}^i = -(\partial_j\pi^{ij} + \partial_j\pi^{ji} + 2\gamma^i{}_{jk}\pi^{jk}).$$

Here $\gamma^i{}_{jk}$ is the spatial Christoffel symbol.

The equations of motion are then derived by the Poisson brackets $\partial_t\cdot = \{\cdot, H\}$ for each of the pairs of conjugate variables ξ,β_i,γ_{ij},π_ξ,π^i, and π^{ij}. These six brackets produce all the constraints and dynamical equations needed except that one must still eliminate some non-dynamical degrees of freedom, namely ξ and β^i. This is addressed more below.

2.2. Kaluza-Klein Theory

Kaluza-Klein theory has historically been used to unify forces with gravity via a disformal relationship, in which case the other forces such as electromagnetism are the result of a choice of slicing in the 4+k-D manifold for k additional dimensions. KK theory has also been used to generate potential dark matter candidates, e.g., bosonic, based on light particles in a compactified dimension[30][31]. The approach taken in this paper proposes that additional fields in Kaluza-Klein theory could be responsible for MOND behavior in galaxies.

Kaluza-Klein theory has already had a major role in dark matter theories. While in this paper, we look primarily at what would be the $n = 0$ modes in a compactified theory (and the only modes in a non-compactified one) $n = 1$ modes allow for the creation of a light particle that is a major contender for dark matter [31]. Thus, combining this theory with that one results in a unified hybrid theory which may be essential to explaining all dark matter-attributed phenomena. (The theory described in this paper could also be combined with mimetic gravity instead which would result in a different hybrid approach which is left for another paper.)

The lightest Kaluza-Klein particle theory depends on a theory of extra dimensions called minimal Universal Extra Dimensions or mUED. In this theory, there are no branes (as in string theory) and Standard Model fields are free to traverse all available dimensions. Momenta is conserved, leading to degenerate KK mode masses. KK number is conserved in the four dimensional effective theory. These are degenerate with masses of n/L where L is the size of the compact dimension. We experience the influence of the compactified dimension from neutral gauge bosons or KK neutrinos. They have no electromagnetic interaction and are non-baryonic. They are also predicted to have a long lifetime. Hence, they are a viable candidate for dark matter.

As mentioned above, however, standard, non-interacting dark matter theories fail to explain all the phenomena we attribute to dark matter, particularly galactic rotation curves. Therefore, it makes sense to look at whether $n = 0$ modes of the KK theory contribute.

To do this, we for now ignore the $n > 0$ modes and focus on the ADM formalism which allows us to produce an effective 4D theory from higher dimensional theory.

For the remainder of the paper, assume $k = 1$ and the additional dimension is $\tau = x_4$ but that does not preclude generalizations to more dimensions.

The application of the ADM formalism requires a time coordinate be specified. In the case of Kaluza-Klein theory we can choose either the usual time coordinate t or the additional dimension τ to be the time coordinate depending on which direction we want to define the evolution of the spatial or spacetime slices. Since we are going to draw an equivalence between the 4-vector potential of generalized TeVeS and the shift function, we must choose τ as our time coordinate here so that the shift function acts like an ordinary 4-vector.

The KK metric obeys the following:

$$dS^2 = -\epsilon\alpha^2 d\tau^2 + g_{ab}(dx^a + \beta^a d\tau)(dx^b + \beta^b d\tau) \tag{4}$$

[32] where $\epsilon = 1$ for timelike and $\epsilon = -1$ for spacelike additional dimension τ.

As described in the previous subsection, the ADM formalism provides all constraints and evolution equations but has some unconstrained degrees of freedom in the choice of lapse function and shift vector. To constrain these is called applying a "slicing condition". This condition describes how the 5D manifold slices into 4D submanifolds and results in coupled equations for the submanifold which in our case is a spacetime metric, g_{ab}, shift 4-vector, β^a, scalar lapse function, α.

In our KK-ADM formalism, the μ function in MOND emerges from the slicing condition. In the ADM slicing conditions are chosen based on practical requirements such as avoiding singularities [33] and removing interdependencies in degrees of freedom [34]. Slicing conditions in 3+1-D GR define the set of spacelike hypersurfaces, $\Sigma_3(t)$, with spatial metric γ_{ij} from one moment in time to the next, where time, t, is a global coordinate parameter. We refer to this condition as spacetime synchronization. The slicing condition enforces what is a standard interval of time at each point in a spatial manifold.

The lapse function which we call α but is sometimes called N is typically a non-dynamical scalar field that one must constrain. Intuitively, the lapse function is the infinitesimal tick interval of time in 4D or in our case in the τ dimension (see Figure 21.2 in MTW [35]). Let the lapse function be a function of μ in order to subsume MONDian physics.

The rest of the condition is fixed by a constraint on the shift vector which defines the degree of coordinate shift from one thin slice to the next with zero shift vector being a normal coordinate system.

In a $4+1$ KK theory, a set of submanifolds $\Sigma_4(\tau)$ including time as a dimension within the slice are defined with metric g_{ab} based on a parameter τ which is analogous to time but may be spacelike. This is referred to as the fifth dimension or the τ dimension.

3. Isham-Kuchař Reference Fluid Theory

The method used in this paper involves a coordinate restricted ADM formalism with a reference fluid. This method was pioneered by Isham, Kuchař [36][36], and Unruh [37] with the goal of establishing a general geometrodynamics approach for arbitrary coordinate restrictions based on the idea that coordinates have physical significance. Isham later applied these ideas to canonical quantum gravity [38].

In standard practice, coordinate restriction is analogous to gauge fixing and, in the general language of dynamical systems, removing non-dynamical variables. All approaches to the ADM formalism need to remove these since the lapse α and shift β^a functions (which Isham uses the symbols N and $\vec{N}$ for respectively) are non-dynamical degrees of freedom that depend only on one's choice of coordinate system.

When the ADM formalism is derived, we arrive at a Hamiltonian $H(\alpha, \beta^a)$ (the details of which can be found in any textbook) and, taking the variation of the Hamiltonian, we arrive at several equations two of which are the dynamical equations for the metric,

$$\dot{g}_{ab}(x,\tau) = \frac{\delta H[\alpha, \beta^a]}{\delta \pi^{ab}(x,\tau)}, \tag{5}$$

and

$$\dot{\pi}^{ab}(x,\tau) = \frac{\delta H[\alpha, \beta^a]}{\delta g_{ab}(x,\tau)} \tag{6}$$

where g and π are the generalized coordinates for the metric and its momentum in phase space. Varying with respect to β^a and α respectively yield the constraints,

$$\mathcal{H}_a(x; g, \pi) = 0, \quad \mathcal{H}(x; g, \pi) = 0. \tag{7}$$

In the standard ADM formalism, the lapse and shift α and β^a are usually non-dynamical degrees of freedom. (Their time derivatives are Lagrange multipliers.)

Two common methods for eliminating the non-dynamical degrees of freedom are:

(i) The first is to set the lapse and shift functions to some functions on $\Sigma_4 \times \mathbb{R}$. These functions are then substituted back into the Hamiltonian equations of the ADM formalism and result (in the case of 4+1-D) five equations of the form $\dot{F}^A(x, \tau) = 0$ for $A = 0, 1, 2, 3, 4$. These have solutions $F^A(x, \tau) = f^A(x)$ for any arbitrary set of functions f^A on Σ_4.

(ii) Restrict the phase space path with five conditions $F^A(x, \tau; g, \pi) = 0$. In this case, care must be taken not to remove genuine physical degrees of freedom. Substituting $\dot{F}^A(x, \tau; g, \pi) = 0$ into 5 and 6 results in five elliptic partial differential equations for α and β^a.

These methods propose, however, that spacetime events are merely marked by artificial coordinate systems that have no physical significance. They can be affected by gravity but cannot affect it.

Alternatively, Isham, Kuchař, and Torre developed a "reference fluid" method for Gaussian [39] and Harmonic [40] coordinates and Unruh for unimodular [37]. In this method, the reference fluid is interpreted as a dynamical physical entity marking off time and/or space. One then attaches these coordinate constraints to the Einstein-Hilbert action using Lagrange multipliers. This can be applied in both the Lagrangian (action) formalism as well as the Hamiltonian above.

Kuchař and Torre applied this method to convert the Wheeler-DeWitt equation into a Schrödinger equation [24] and it has general applicability to canonical quantum gravity. In their approach, spacetime events are provided by a reference system that interacts with gravity. In other words, not only does it measure events, but it is a material system coupled to gravity present in the Einstein-Hilbert action from the beginning of the derivation rather than being applied at the end as in the standard approach. Similar approaches are well-known in quantum gravity going back to DeWitt who coupled gravity to an elastic medium carrying mechanical clocks [41].

As an example of how this method differs from the standard ADM, as detailed in [40], when the Harmonic condition is applied to the Einstein-Hilbert action, a set of fields λ_a and X_a are defined which represent the Lagrange multipliers and harmonic coordinates respectively. The Harmonic condition is $\gamma^{AB}\nabla_A\nabla_A X^A = 0$. The action is defined as $S[\gamma, \lambda] = S^E[\gamma] + S^F[\gamma, \lambda]$ where S^E is the Einstein-Hilbert action and

$$S^F = -\frac{1}{2}\int |\gamma|^{\frac{1}{2}}\gamma^{AB}\nabla_A\lambda_C\nabla_B X^C.$$

The variation yields, $G^{AB} = \frac{1}{2}T^{AB}$ where $T^{AB} = 2|\gamma|^{-\frac{1}{2}}\frac{\delta S^F}{\delta\gamma_{AB}}$. Thus, the coordinate restriction has gravity interacting with massless scalar fields with stress-energy tensor

T^{AB} but these fields depend only on the coordinates and Lagrange multipliers. In the unimodular case $|\gamma| = 1$, the resulting stress-energy tensor looks like a cosmological constant [37].

One can eliminate these additional sources by only looking at phase space paths such that the Lagrange multipliers $\lambda_A = 0$ or, as in [40], consider the extended phase-space where Lagrange multipliers are included to enforce the gauge condition. This approach is broadly applicable to coordinate conditions with reference fluids.

The physical interpretation is that the Isham-Kuchař representation theory is equivalent to the coupling of gravity to reference systems that represent privileged space and time coordinates. Massless scalar fields provide the privileged space and time reference system. In other words, clocks and position markers become physical fields that couple to gravity itself. For example, in the Gaussian coordinate case [24], spacetime events are identified via the velocity potentials of a heat conducting fluid.

As demonstrated in the next section, this approach is used but only to affect the τ coordinate as Kuchař does for time [26]. The other four coordinates are freely covariant.

4. Derivation of TeVeS Theory from Kaluza-Klein

This TeVeS theory [42] gives the action,

$$S_{\text{TeVeS}} = \frac{1}{16\pi G}\int d^4x\,\sqrt{\hat{g}}\Big[R - \hat{K} + \lambda_\beta(\beta^a\beta_a + 1) - \mu g^{ab}\partial_a\phi\partial_b\phi - V(\mu)\Big] + S_M[\tilde{g}]. \tag{8}$$

where

$$\hat{K} = \hat{K}^{cdab}\nabla_c\beta_d\nabla_a\beta_b \tag{9}$$

and

$$\hat{K}^{cdab} = c_1 g^{ca}g^{db} + c_2 g^{cd}g^{ab} + c_3 g^{cb}g^{da} + c_4 g^{db}\beta^c\beta^a. \tag{10}$$

The original TeVeS theory of Bekenstein [16] assumed that c_i were constant, which caused spin-2 tensor modes of GW to not match the propagation of electromagnetic modes [17]. Recent observations of GW170817 by LIGO require the propagation speed of the gravitational tensor c_T to satisfy constraint of $|c_T - 1| < 10^{-15}$ in units where the speed of light in vacuum $c = 1$ [19]. Skordis et al. [17] show that in TeVeS theory, if $c_{13} = c_1 + c_3$, then,

$$c_T = e^{-4\phi}/(c_{13} - 1) \tag{11}$$

If $c_T = 1$, this implies that,

$$c_{13} = 1 - e^{-4\phi}. \tag{12}$$

The theory can be derived starting from a $D = 4 + 1$ Kaluza-Klein formalism using the Lagrangian (action) method.

Let the non-conformally scaled, Jordan metric‡ have the form:

$$\gamma'_{44} = 1/\alpha^2 + g'_{ab}\beta^a\beta^b, \tag{13}$$

$$\gamma'_{4a} = \beta_a, \quad \gamma'_{ab} = g'_{ab}, \tag{14}$$

and the inverse is:

$$\gamma'^{44} = \alpha^2, \tag{15}$$

$$\gamma'^{4a} = \alpha^2\beta^a, \quad \gamma'^{ab} = g'^{ab} + \alpha^2\beta^a\beta^b, \tag{16}$$

This formulation of KK is non-standard from the textbook version, e.g., see [43], because it reverses the covariant and contravariant forms from the standard and inverts γ'_{44}. In addition, we do not equate β^a with the electromagnetic vector potential since it is needed for the gravitational theory. Thus, the formulation is equivalent to the formalism [27] but in one additional dimension, and ADM vocabulary is used such as referring to the β^a as the "shift vector" and α as the "lapse function" below. Note: the lapse function α here is the inverse of the lapse function $\xi = 1/\alpha$ in the section above.

Let $\alpha = e^{-3\phi}$. Define the Pauli metric, $\tilde{\gamma}_{AB}$, in terms of the Jordan metric: $\gamma'_{AB} = \alpha^{2/3}\tilde{\gamma}_{AB} = e^{-2\phi}\tilde{\gamma}_{AB}$.

$$\tilde{\gamma}_{44} = e^{8\phi} + \tilde{g}_{ab}\beta^a\beta^b, \tag{17}$$

$$\tilde{\gamma}_{4a} = e^{2\phi}\beta_a, \quad \tilde{\gamma}_{ab} = \tilde{g}_{ab}. \tag{18}$$

The KK action, under the conformal relationship between $\tilde{\gamma}$ and γ' is,

$$\begin{aligned} \mathcal{S}_{KK} &= \frac{1}{16\pi G}\int d^5x\,\sqrt{-\gamma'}{}^{(5)}R \\ &= \frac{1}{16\pi G}\int d^5x\,\sqrt{-\tilde{g}}\left({}^{(5)}\tilde{R} - \frac{16}{3}e^{\frac{3\phi}{2}}\Delta e^{-\frac{3\phi}{2}}\right), \end{aligned} \tag{19}$$

where Δ is the Laplace-Beltrami operator in 5-D.

This eliminates the dependence on α in the volume element from the action.

The Jordan metric's submanifold components have the form,

$$\gamma'^{ab} = e^{2\phi}\tilde{g}^{ab} + e^{-4\phi}\beta^a\beta^b. \tag{20}$$

We consider this to be the "physical" metric while the Pauli is the gravitational. A test mass with velocity vector u_A such that,

$$\gamma'^{AB}u_Au_B \tag{21}$$

‡ The nomenclature used in this paper is the Jordan metric versus the Pauli metric and has been in use for many decades in that community [43]. Hill and Ross also use the term Jordan frame versus Einstein frame which can cause some confusion here because they show there must be "contact terms" when you have non-minimal coupling to the scalar gravitational field [44]. These terms are point interactions that would occur when you have non-minimal coupling. The term Jordan metric is simply a name for an unscaled metric but has a flipped meaning from Hill and Ross here when you get into the details. In this work, the derivations starts in what Hill and Ross call the "Einstein frame" and make a Weyl transformation to what they call the Jordan frame. The reference fluid interactions is then added to the Jordan frame. Transforming back to the Einstein frame would reveal the contact terms.

defines its geodesic. We neglect u_4 here which would imply propagation in the extra dimension. In this case, the geodesic is defined by Bekenstein-Sanders (1).

The KK action in the cylinder condition, which is enforced by the compactification, where all derivatives with respect to x_4 are zero, obeys the following relationship as well,

$$\begin{aligned} S_{KK} &= \frac{1}{16\pi G}\int d^5x\,\sqrt{-\gamma'^{(5)}}R' \\ &= \frac{1}{16\pi G}\int d^4x\,\frac{\sqrt{-g'}}{\alpha}\left({}^{(4)}R' + \frac{\alpha^2}{4}F'_{ab}F'^{ab}\right), \end{aligned} \tag{22}$$

where $F_{ab} = \nabla_a\beta_b - \nabla_b\beta_a$. This is before applying the conformal transformation 19. This action simply is another way of writing the ADM action (2) where you can show that $\frac{\alpha^2}{4}F_{ab}F^{ab} = K_{ab}K^{ab} - K^2$ applying the definition of extrinsic curvature (3).

Making the conformal transformation (19) to eliminate the lapse function from the volume element, the action is,

$$\begin{aligned} \mathcal{S}_{KK} &= \frac{1}{16\pi G}\int d^4x\,\sqrt{-\tilde{g}} \\ &\left(K^{cdab}\nabla_c\beta_d\nabla_a\beta_b + {}^{(4)}R - \frac{16}{3}e^{\frac{3\phi}{2}}\Delta e^{-\frac{3\phi}{2}}\right) \end{aligned} \tag{23}$$

where $K^{cdab} = e^{-6\phi}(\frac{1}{2}\tilde{g}^{ac}\tilde{g}^{bd} + \frac{1}{2}\tilde{g}^{bc}\tilde{g}^{ad} - \tilde{g}^{ab}\tilde{g}^{cd})$.

Using $\Delta = \nabla^a\partial_a$ where $\nabla^a v_a = g^{ab}[\partial_b v_a + \Gamma^c{}_{ab}v_c]$,

$$e^{3\phi/2}\Delta e^{-3\phi/2} = -\frac{3}{2}\nabla^a\partial_a\phi + \frac{9}{4}\partial^a\phi\partial_a\phi.$$

The total covariance term, first term on the RHS, does not contribute to the equations of motion.

The shift functions are timelike $\beta^a\beta_a = -\beta^2$ where

$$\beta^2 = e^{6\phi} - e^{2\phi} \tag{24}$$

is a scalar function. This ensures that the universally coupled metric 20 has the same form as the Bekenstein-Sanders (1) since one may replace the vector with $A^a = \beta^a/\beta$ and pull out the factor of β^2.

The constraint has auxiliary field λ_β, and the action is now,

$$\begin{aligned} \mathcal{S}_{KK\lambda} = \frac{1}{16\pi G}\int d^4x\,\sqrt{-\tilde{g}}\Big(&K^{cdab}\nabla_c\beta_d\nabla_a\beta_b + \\ &{}^{(4)}R - 12\partial^a\phi\partial_a\phi + \lambda_\beta(\beta^a\beta_a + \beta^2)\Big). \end{aligned} \tag{25}$$

We can also write the term $K^{cdab}\nabla_c\beta_d\nabla_a\beta_b = e^{-6\phi}\frac{1}{2}g^{ac}g^{bd}F_{ab}F_{cd}$.

Add an auxiliary field μ_1. This has a separate action,

$$S_\mu = -\frac{1}{16\pi G}\int d^4x\,\sqrt{-\tilde{g}}\left[\mu_1\hat{g}^{ab}\partial_a\phi\partial_b\phi + U(\mu_1)\right], \tag{26}$$

where μ_1 is a dimensionless scalar field and

$$\hat{g}^{ab} = \tilde{g}^{ab} - c_4\beta^a\beta^b, \tag{27}$$

where c_4 is the constant in 10. The potential $U(\mu)$ is an arbitrary function that depends on a scale l_B.

The auxiliary fields λ_β and μ_1 are constraints on the shift vector β^a and scalar field ϕ but also represent the reference fluid. Their function is to orient and scale the $3+1$-D slices in the fifth dimension based on a broken displacement symmetry in the τ dimension.

The connection between the conformal rescaling and the slicing condition is a consequence of degrees of freedom in the metric. While the actions are all in terms of the unscaled Jordon metric, $\tilde{\gamma}$, rather than the rescaled Pauli metric, γ [43], the Jordan metric is not completely free for a choice of conditions. In numerical simulations of relativity using the ADM formalism, e.g., York's conformal dynamics slicing [34] and thin sandwich model [45], the conformal scale of the submanifold cannot be chosen arbitrarily but must be solved for given a choice of slicing. The same is true in higher dimensions. Hence, only the conformally invariant aspect of the submanifolds have unconstrained degrees of freedom. This suggests that slicing and conformal freedom are not separate degrees and that the Pauli metric representation is needed.

We now want to take the variation of S_{KK} with respect to the three fields ϕ, β^a, and g_{ab}. Write the action as a sum of actions. Let $S_{KK} = S_R + S_\beta + S_\phi$. Then the full action with the constraints is one for each term, $S_{KK\mu\lambda} = S_R + S_\beta + S_\phi + S_\lambda + S_\mu$.

We assume that matter is given by some Lagrangian dependent on a variety of fields $S_M = \int d^4x\,\sqrt{\gamma'}\mathcal{L}_M[\gamma'^{ab}, \beta^a, \phi, \Psi, A^a]$. In this section, we are only concerned with the stress-energy tensor $\delta S_M = -\frac{1}{2}\int d^4x\sqrt{-g'}T_{ab}\delta g^{ab}$ where $T_{4a} = T_{44} = 0$.

Variation of $S_{KK\mu\lambda} + S_M$ with respect to the metric $\tilde{g}^{ab}$ leads to the field equations,

$$\begin{aligned}\tilde{G}_{ab} = & -e^{-6\phi}(\tilde{g}^{cd}F_{ca}F_{db} - \frac{1}{4}\tilde{g}_{ab}F^{cd}F_{cd}) + \\ & (12+\mu_1)\partial_a\phi\partial_b\phi + 6\tilde{g}_{ab}(U' - \beta^a\beta^b\partial_a\phi\partial_b\phi) \\ & -2\beta^c\partial_c\phi\beta_{(a}\partial_{b)}\phi - \lambda_\beta\beta_a\beta_b + \\ & 8\pi G[T_{ab} + 2e^{-6\phi}\beta^c T_{c(a}\beta_{b)}] + \frac{1}{2}(\mu U' - U)\tilde{g}_{ab}\end{aligned} \tag{28}$$

and we have used the slicing condition,

$$\hat{g}^{ab}\partial_a\phi\partial_b\phi = -U' \tag{29}$$

and $U' = \partial U/\partial\mu_1$.

For variations with respect to ϕ and β^a, one can apply the Euler-Lagrange and rearranging,

$$\begin{aligned}&\nabla_a([(12+\mu_1)\tilde{g}^{ab}\partial_b\phi + \mu_1\beta^a\beta^b\partial_b\phi) = \\ &8\pi G\left(\tilde{g}^{ab} + e^{-6\phi}\beta^a\beta^b\right)T_{ab} + 6K^{cdab}\nabla_c\beta_d\nabla_a\beta_b\end{aligned} \tag{30}$$

The final equation for the vector field is,

$$\begin{aligned}\nabla_c(e^{-6\phi}F^c{}_a) = & -2\lambda_\beta\beta_a - \mu_1\beta^c\partial_c\phi\partial_a\phi + \\ & (8\pi G)e^{-6\phi}\beta^c T_{ca},\end{aligned} \tag{31}$$

which, by contracting with β^a, solves for λ_β,

$$\lambda_\beta = \frac{1}{2}\Big[-\beta^a\nabla_c(e^{-6\phi}F^c{}_a) - \mu_1\beta^a\beta^c\partial_a\phi\partial_c\phi - 8\pi G e^{-6\phi}\beta^a\beta^c T_{ac}\Big](e^{6\phi} - e^{2\phi})^{-1} \quad (32)$$

Thus, we have a total of 15 field equations.

One can easily recover general equivalence to 3+1-D GR by rescaling $\lambda_\phi\phi' = \phi$ and $\lambda_\phi \to 0$, in which case both the scalar and vector parts of the action go to zero. This is also equivalent to letting $\alpha^2 \to 0$ in the original Kaluza-Klein theory.

4.1. Quasi-static limit

Both Newtonian and MOND behavior are recovered in the case of slow motion and weak potentials. In the quasi-static limit, one expands $\phi = \phi_0 + \varphi$ where ϕ_0 is constant and φ is independent of time and $|\varphi| \ll 1$. The Einstein-Hilbert metric becomes $g_{00} = e^{-2\phi_0}(1 - 2\Psi)$ and $g_{ij} = e^{2\phi_0}(1 - 2\Theta)\gamma_{ij}$ where $\gamma_{ij} = \delta_{ij} + h_{ij}$. The shift vectors are $\beta_0 = -e^{-\phi_0}(e^{6\phi_0} - e^{2\phi_0})(1 + \Psi)$, $\beta^0 = -e^{\phi_0}(e^{6\phi_0} - e^{2\phi_0})(-1 + \Psi)$ and $\beta^i = \beta_i = 0$. The matter metric is then $\tilde{g}_{00} = -(1 - 2\tilde{\Psi})$ and $\tilde{g}_{ij} = (1 - 2\tilde{\Phi})\gamma_{ij}$ where $\tilde{\Psi} = \Psi - \varphi$ and $\tilde{\Phi} = \Phi - \varphi$.

The quasi-static limit has the same assumptions as the first Parameterized Post-Newtonian (1PPN) limit. The gravitational field is a small fluctuation about the background Minkowski spacetime. Matter is represented with an effective perfect fluid with density ρ, pressure p, internal energy Π and 3-velocity $\vec{v}$. All fields are expanded perturbatively in orders of $v = |\vec{v}|$. Let Φ_ρ be the Poisson potential from the baryonic only matter density $\vec{\nabla}^2\Phi_\rho = 4\pi G_N\rho$ and G_N Newton's constant. As in the PPN formalism, $\partial/\partial t \sim O(v)$, $\Phi_\rho \sim \rho \sim \Pi \sim \varphi \sim O(v^2)$, $p \sim O(v^4)$. We also have $h_{ij} \sim O(v^2)$ and $\beta_i \sim O(v^3)$.

The quasi-static limit only contains terms up to $O(v^2)$ so terms containing p and $\rho\Pi$ are ignored. This means that matter is simply dust $T_{ab} = \rho u_a u_b$ with a normalized four velocity u^a.

With these assumptions, the spatial part of the equations 28 reduces to $G_{ij} = 0$ as in TeVeS. The disformal transformation gives $h_{ij} = 2[\varphi - \gamma_{PPN}\Phi_N]\delta_{ij}$ and shows that $\gamma_{PPN} = 1$. All the other terms are $O(v^3)$ or greater except those corresponding to the MOND equations [42][17]. Let $\mu = \mu_1 + 12$. Then,

$$\nabla^2\tilde{\Psi} = \frac{8\pi G}{2 - c_1 + c_4}\rho, \quad (33)$$

$$\nabla_i(\mu\nabla^i\varphi) = 8\pi G\rho, \quad (34)$$

$$\tilde{\Phi} = \tilde{\Psi} \quad (35)$$

where in this case $c_1 = \frac{1}{2}(1 - e^{-4\phi_0})$ and c_4 is chosen in the constraint 27.

Define $\partial_\perp = (\partial_4 - \mathcal{L}_\beta)$, where $\mathcal{L}_\beta$ is the Lie derivative along the shift vector, as a derivative perpendicular to the 4D submanifold.

For a scalar field, $\mathcal{L}_\beta = \beta^a\partial_a\phi$, and, for example, Bona-Massó slicing imposes a condition on the the lapse function, $\partial_\perp\alpha = -\alpha^2 a(\alpha)K$ where K is the trace of the extrinsic curvature and a is a function that gives the desired slicing. Slicing conditions, in general, are expressed as an equation involving a derivative of how the lapse function changes perpendicular to the submanifold, $\partial_\perp \alpha = f(\alpha, \partial_a \alpha)$.

Given that we assume $\partial_4 \cdot = 0$, and our MOND slicing condition (29) can be rewritten,

$$\partial_\perp \phi = \pm \sqrt{\frac{dU}{d\mu} + \partial^a \phi \partial_a \phi} \tag{36}$$

Since $\phi = -\frac{1}{3} \log |\alpha|$ and

$$\partial_a \phi = \frac{1}{3\alpha} \partial_a \alpha$$

we can make a substitution so that this becomes a condition on the lapse function directly. If we let

$$U = \left(\frac{1}{3\alpha} \right)^2 U_\alpha \tag{37}$$

then the condition on the lapse function is

$$\partial_\perp \alpha = \pm \sqrt{\frac{dU_\alpha}{d\mu} + \partial^a \alpha \partial_a \alpha}. \tag{38}$$

This is now written as a slicing condition perpendicular to the submanifold.

Equation 38, combined with the condition on β^a, 24, now fixes the 5D theory.

To quasi-static order, $U \sim V$, where V is the standard TeVeS potential given in 8. MOND defines μ as $\mu = \frac{df}{dX}$ where $X = l_B^2 \hat{g}^{ab} \partial_a \phi \partial_b \phi$ and $f = \mu X + l_B^2 V$. It does not precisely specify μ or V; it only determines two limits. MOND is achieved if,

$$\frac{dV}{d\mu} \to -\frac{4}{9 b_0^2 l_B^2} \mu^2$$

where b_0 is a constant determined by ϕ_0 and the MOND acceleration parameter a_0 [42]. Meanwhile, it diverges for $\mu \to \mu_0$, for example, $\frac{dV}{d\mu} \to (\mu_0 - \mu)^{-m}$, gives the Newtonian limit where μ_0 is a constant.

Since μ is a function of X, this means that perpendicular to the submanifold in the Newtonian limit,

$$\partial_\perp \phi \approx \pm \sqrt{\frac{dV}{d\mu}} \to \infty.$$

This is true if $\alpha \to 0$ according to 37 which is consistent with standard General Relativity (the vector potential drops out of the equations). This occurs if the scalar field is very large, $\phi \to \infty$.

In the MOND limit, on the other hand, the slicing condition is

$$\partial_\perp \phi \to \pm \sqrt{\partial^a \phi \partial_a \phi - \frac{4}{9 b_0^2 l_B^2} \mu^2} \tag{39}$$

where under spherical symmetry, $\mu \to \frac{2G_N}{G} \frac{1}{l_B a_0} e^{\phi_0} \sqrt{l_B^2 \hat{g}^{ab} \partial_a \phi \partial_b \phi}$ and G_N depends on μ_0 and ϕ_0 [42].

Once the degrees of freedom are selected, the scalar field's spacetime distribution is determined by 30 with the primary influence being the baryonic matter distribution. Its evolution in the 5th dimension is determined by the slicing condition 36.

The slicing condition is the source of MOND phenomenon. The equation 36 is in terms only of scalar quantities with respect to the 3+1-D manifold. Thus, it is invariant under 3+1-D coordinate transformations.

5. FLRW Backgrounds and Spin-2 Tensor Propagation

TeVeS requires fine tuning (12) that ensures that it matches empirical data against Friedmann-Lemaître-Robertson-Walker (FLRW) backgrounds. In particular, the tensor mode propagation speed and Shapiro delay must be the same as electromagnetic waves to agree with observations of GW170817 [28]. This fine tuning is *ad hoc* in TeVeS but predicted in the KK theory.

The prediction comes from the unique determination of the Bekenstein-Sanders metric (1). Sanders showed that the metric's form (up to a constant scaling of the scalar field) is determined by global conformal symmetry and the independence of units of the fine structure constant [46]. Indeed, the form is uniquely determined. By enforcing the form of the metric to agree with this, however, the KK theory automatically predicts the fine tuning used in [17].

The key observation is in the differences between the equations for TeVeS (as derived from 8 and given in [17]) and those of the KK theory's equation for the Einstein tensor (28).

Let the universally coupled metric g_{ab} have the following form,

$$ds^2 = -dt^2 + a^2(\gamma_{ij} + \chi_{ij})dx^i dx^j. \tag{40}$$

In this case, a is the scale factor, γ_{ij} is the spatial metric of constant curvature κ, and χ_{ij} is the transverse-traceless tensor mode GW such that $\gamma^{ij}\chi_{ij} = 0$ and $\nabla_i \chi^i{}_j = 0$.

In the case of tensor modes, ϕ and the shift β^a are unperturbed, $\beta^i = 0$, and $\phi = \bar{\phi}(t)$, the spatial derivatives in ϕ drop out.

To show that the Kaluza-Klein equations for tensor mode propagation are the same as TeVeS. We could go through the lengthy and tedious computations which can be found in [47], or we exploit a property of FLRW spacetime to show equivalence trivially. The later approach gets around the fact that the KK shift vector has an additional factor dependent on the lapse function not present in TeVeS. The key is to remove any derivatives of β^a from the action (25) for FLRW backgrounds in which case the factor can be pulled out as shown in the following:

By construction, we have β^a timelike orthogonal to the 3D spatial hypersurface. This is a non-trivial affinely parameterized geodesic vector field. By the Frobenius theorem, the twist tensor is naught: $F_{ab} = \nabla_{[a}\beta_{b]} = 0$. Therefore, terms dependent on F_{ab} drop out of the action (25). Hence, the action is only left with terms that have no derivatives of the shift vector β^a. Now, replace the time oriented component β^0 with

$(e^{6\bar{\phi}} - e^{2\bar{\phi}})A^0$ where A^0 is the time component of the TeVeS vector field. The factor of $(e^{6\bar{\phi}} - e^{2\bar{\phi}})$ multiplies with the factors of $e^{-6\bar{\phi}}$ from the lapse function in the action to give a factor $1 - e^{-4\bar{\phi}}$ which matches the factor in the TeVeS action [17]. Thus, in the perturbation theory of spin-2 tensor modes against an FLRW background, the two theories have the same action.

Let $A_0 = -e^{-\bar{\phi}}$. This means that the tensor mode equations are the same in both theories in the perturbation theory. The tensor mode obeys [17],

$$e^{2\bar{\phi}}(1 - c_{13})[\ddot{\chi}^i{}_j + (3H + 4\dot{\bar{\phi}})\dot{\chi}^i{}_j] - e^{2\bar{\phi}}\frac{dc_{13}}{d\phi}\dot{\bar{\phi}}\dot{\chi}^i{}_j - \frac{1}{a^2}e^{-2\bar{\phi}}(\nabla^2 - 2\kappa)\chi^i{}_j = 16\pi G e^{-2\bar{\phi}}\Sigma^{(g)i}{}_j. \tag{41}$$

Here $H = \dot{b}/b$ is the rescaled Hubble parameter where $b = ae^{\phi}$ and $\Sigma^{(g)i}{}_j$ is a traceless matter term related to anistropic stress [47]. From 11, $c_{13} = 1 - e^{-4\phi}$ ensures the correct fine tuning. For the KK theory, the product of the square lapse function and the square magnitude of the shift gives this value $-\alpha^2\beta^2 = c_{13}$. This is precisely the square of the magnitude of a shift of a point x^a for a temporal distance $\alpha d\tau$ in the fifth dimension required by the Bekenstein-Sanders form of the metric.

This means that, while in TeVeS the value of c_{13} is chosen empirically as a free function, in the KK theory c_{13} is not free at all but fixed by the Einstein-Hilbert action. Nevertheless, they are the same in both theories.

6. Discussion

The MOND condition is quite specific and, if it did not have the required limits for Newtonian and MOND, the rotational curves of galaxies would not match MOND, at least without some additional dark matter components. The Newtonian condition is quite reasonable as it corresponds to points on the 4D slices sitting on a horizon with $\alpha = 0$. This is the case even in the absence of a reference fluid. By analogy, an infalling beam of light would follow such a trajectory in ingoing Eddington-Finkelstein coordinates at the event horizon. (Gullstrand-Painlevé coordinates are a similar choice.) In this case, however, the “ingoing” dimension is not radial but temporal since the shift function points primarily in the time dimension (though not entirely) with the most significant cross term in the Kaluza-Klein metric being $\tau - t$. Thus, one can make an analogy between (1) the event horizon of a black hole with an event horizon which for an infalling observer has $t - r$ cross terms in the metric to (2) a horizon where an “infalling” observer has cross terms in $\tau - t$. The MOND condition, meanwhile, corresponds to an alternative state of motion such as a trajectory slightly outside and perhaps leading away from the horizon. This condition (being pushed away from the horizon) is enforced by the reference fluid which likely arises from symmetry breaking at the quantum level. A thorough investigation of this symmetry breaking is beyond the scope of this paper, but the connection between the MOND condition and symmetry breaking is a prediction of the theory about quantum gravity, and it would be interesting to explore that idea in future research.

Since slicing is simply a choice of coordinates, it comes down to an observer's state of motion relative to the extra dimension in addition to the symmetry breaking that this dimension has (e.g., from spontaneous compactification). This is given by the metric 4. Choose a point in spacetime x^a. A distant observer, O, measures a field as it propagates from τ to $\tau + d\tau$. The observer sees that its location in spacetime, x^a shifts by the shift vector $x^a + \beta^a d\tau$. In addition, compared to proper time, time propagates at a rate of α. For every infinitesimal clock tick in proper time, the field propagates at $x^a + \alpha\beta^a d\tau$. Thus, the slicing is a property of how fields propagate within the extra dimension and that in turn depends on a combination of the distribution of matter and the interpolating function, μ.

Since the slicing condition is analogous to a gauge fixing, it means that parameters such as a_0 and the form of μ may not be fixed as well. This may lead to a 5D Einstein-Hilbert interpretation of a more complex 4D effective theory such as Modified Gravity (MOG) where μ is dependent on the size of the system, e.g., wide binaries or galaxies [48]. This is another subject for a future paper.

The vector field is also analogous to the gravitomagnetic potential in gravitoelectromagnetism or GEM. GEM is well understood when applied to 4-D GR and the same principles can be applied in 5D [49]. The gravitoelectric field from the lapse function, $G^a = -\nabla^a \phi$, for example produces MOND effects while the Newtonian force comes from the time-time component of the metric. Meanwhile, the lensing is due to non-linear variations in the shift vector. The Sagnac effect is an example of a well understood effect on light that comes directly from a shift vector in 4D GR. Tidal forces, differential dragging, described by the gravitomagnetic tidal tensor $\mathbb{H}_{AB} \sim \partial_a \partial_b \beta_c$ may also occur.

Ordinary post-Newtonian GEM cannot explain galactic rotation curves [50][51], despite some flawed attempts to do so [52], because their contributions are order $(v/c)^2$. But higher dimensional GEM can because it can generate much stronger gravitational effects on the same order as the Newtonian force.

This suggests possible tests for the theory involving higher dimensional gravitomagnetic effects similar to the Lense-Thirring and tidal effects that dark matter would not cause.

7. Conclusion

In conclusion, it has been shown that $n = 0$ modes of Kaluza-Klein theory in 5D can replicate generalized TeVeS theory without ruling out additional sources of dark matter from minimal Universal Extra Dimensions (mUED) Kaluza-Klein dark matter Weakly Interacting Massive Particles (WIMPs). It has also been shown that the KK theory constrains most of the freedom in the definition of TeVeS. The only freedom is in the slicing condition which is responsible for MOND and must be matched to empirical data. More complex theories with non-invertible disformal metrics such as mimetic gravity can lead to mimetic dark matter, a potential alternative to dust-like dark matter that does not require specific constraints on KK compactification [53].

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